

# Parity realization in vector-like theories from fermion bilinears

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**ABSTRACT:** We reconsider in this paper the old aim of trying to understand if the observed realization of discrete symmetries as Parity or  $CP$  in the  $QCD$  vacuum can be satisfied from first principles. We show how under the appropriate assumptions implicitly done by Vafa and Witten in their old paper on parity realization in vector-like theories, all parity and  $CP$  odd operators constructed from fermion bilinears of the form  $\bar{\psi}\tilde{O}\psi$  should take a vanishing vacuum expectation value in a vector-like theory with  $N$  degenerate flavours ( $N > 1$ ). In our analysis the Vafa-Witten theorem on the impossibility to break spontaneously the flavour symmetry in a vector-like theory plays a fundamental role.

**KEYWORDS:** Lattice Gauge Field Theories, Spontaneous Symmetry Breaking, Discrete and Finite Symmetries.

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## Contents

<b>1. Introduction</b>	<b>1</b>
<b>2. The Vafa-Witten result: an alternative way</b>	<b>2</b>
<b>3. The <i>p.d.f.</i> for fermionic local operators</b>	<b>5</b>
<b>4. Vacuum expectation values of <math>P = -1</math> fermion bilinears</b>	<b>7</b>
<b>5. Conclusions</b>	<b>9</b>

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## 1. Introduction

The realization of symmetries in Nature is an essential point to understand all the phenomenology of elementary particles. Whereas space-time rotational invariance seems to be always realized, this is not the case for the discrete parity,  $P$ , the charge conjugation,  $C$ , and the time reversal,  $T$ , symmetries.  $P$  and  $C$  symmetries are violated by electroweak processes and the only fundamental theoretical restriction comes from the  $CPT$  theorem, which forbids a violation of the  $CPT$  symmetry in relativistic field theory, even by the vacuum or ground state.

On the other hand the large amount of experimental results on the strong interaction processes strongly suggests that  $P$  and  $C$  are preserved by this interaction, vacuum included. The extremely small experimental bound on the neutron dipolar electric moment suggests that a  $CP$  violating topological term should be excluded from the  $QCD$  Lagrangian; and this result, combined with the solution of the  $U(1)$  problem via the axial anomaly has generated the well known strong  $CP$  problem [1]. Hence the understanding of the realization of these symmetries in QCD is one of the challenges of Theory physicists.

One important ingredient of the fermion gauge interaction in  $QCD$ , as opposite to the electroweak interaction, is the fact that it is vector-like. Years ago Vafa and Witten [2] *conjectured* that parity should not be spontaneously broken in any vector-like theory as, for instance,  $QCD$ .

Although the Vafa and Witten result has been taken for a long time by the scientific community as a theorem, we call it conjecture for the following two reasons:

*i.* V.A. and A. Galante have shown in [3] that in the Vafa-Witten work there is an implicit assumption: the Euclidean QCD Lagrangian in presence of an external source coupled to any  $P = -1$  local pure gluonic operator is a well defined quantum system, or at least a well defined statistical system, but, as a rule, this needn't be true and the system may not be well defined. This assumption may seem innocuous, as suggested in [4], or

even in [5], where the authors argued that these ill-defined theories are not physical, but we would like to stress that at least for a very simple statistical system, the two-dimensional Ising model within an imaginary external magnetic field, the theorem does not hold. Indeed as it was shown in [3], if parity were spontaneously broken in QCD, the theory in presence of an external local  $P = -1$  pure gluonic operator would be ill-defined. Similar concerns on the validity of the Vafa-Witten theorem were raised in [6], and although in [7] some arguments against [3] were exposed, the author claimed to have disproved the controversial theorem.

*ii.* There are explicit counter-examples for operators built from fermion fields, as the parity-flavour breaking term involved in the Aoki phase [8] for Wilson fermion lattice QCD. Even though in the Vafa-Witten original paper it is stated in a foot note that the extension to fermionic operators, after the integration of fermionic degrees of freedom, is straightforward; this is not true: their argument can not be extended to order parameters constructed from fermion fields. This point, which has been stressed in [9], will also be discussed in this paper.

The aim of this paper is to clarify a little bit these points and, mainly, to extend the Vafa-Witten conjecture to fermion bilinear order parameters, under suitable hypothesis.

In section 2 we briefly review the Vafa-Witten argumentation, and give an alternative way to reproduce their result, with the help of the probability distribution function (hereafter *p.d.f.*) of the order parameter. We discuss also in this section why the Vafa-Witten result can not be extended in a simple way to order parameters constructed from fermion bilinears. Since our extension of Vafa-Witten result to fermion bilinears is based in the *p.d.f.* of these operators, we give also a short review, in section 3, of the formalism developed in [10] to define the *p.d.f.* of fermionic operators. In section 4 we apply this formalism to the extension of the Vafa-Witten result to fermion bilinear order parameters, always under the assumption that a quantum theory can be consistently constructed in presence of these operators in the theory lagrangian. Section 5 contains our conclusions.

## 2. The Vafa-Witten result: an alternative way

The Euclidean partition function of a vector-like gauge theory can be written as the following path integral over the Grassmann and gauge degrees of freedom,

$$\mathcal{Z} = \int [dA_\mu^a][d\bar{\psi}d\psi] \exp\{-S_{\text{YM}} + \bar{\psi}(\not{D} + m)\psi\} \quad (2.1)$$

where  $S_{\text{YM}}$  is the action for the Yang-Mills fields and the gauge-fermion coupling is a bilinear of the fermion fields.  $\not{D}$  is the Euclidean Dirac operator and what characterizes vector-like theories is that the fermion determinant  $\det(\not{D} + m)$ , which appears in the effective gauge theory after integration of the fermion fields, is positive-definite for any non vanishing bare fermion mass.

If we add now to the action a hermitian  $P = -1$  local order parameter  $Y(A_\mu^a)$ , which depends only on the gauge fields, as for instance a topological  $\theta$ -vacuum term, the Euclidean

partition function for the effective gauge theory is:

$$\mathcal{Z} = \int [dA_\mu^a] \exp \left\{ - S_{\text{YM}} - \lambda \int d^4x Y(x) + \ln \det(\not{D} + m) \right\} \quad (2.2)$$

Years ago Vafa and Witten [2] gave an argument against spontaneous  $P$  breaking in vector-like parity-conserving theories, as  $QCD$ . The main point in their argument was the observation that any arbitrary hermitian local order parameter,  $Y(A_\mu^a)$ , constructed only from Bose fields, should be proportional to an odd power of the four indices totally antisymmetric tensor  $\epsilon^{\mu\nu\rho\sigma}$  and therefore it would pick up a factor of  $i$  under Wick rotation. The local order parameter  $Y(A_\mu^a)$  can therefore be written as  $Y(A_\mu^a) = iX(A_\mu^a)$  with  $X$  real, and the Euclidean partition function (2.2) becomes:

$$\mathcal{Z} = \int [dA_\mu^a] \exp \left\{ - S_{\text{YM}} - i\lambda \int d^4x X(x) + \ln \det(\not{D} + m) \right\} \quad (2.3)$$

The addition of an external symmetry breaking field  $\lambda Y(A_\mu^a)$  to the Lagrangian in Minkowski space becomes then a pure phase factor in the path integral definition of the partition function in Euclidean space (2.3). But a pure phase factor in the integrand of a partition function with positive-definite integration measure can only increase the vacuum energy density (free energy density). Vafa and Witten concluded that, in such a situation, the mean value of the order parameter should vanish in the limit of vanishing symmetry breaking field.

The only objection done in [3] was that any argumentation on the vacuum energy density (free energy density) of the theory in presence of a parity breaking term  $\lambda$  requires the previous assumption that the quantum theory is well defined in such a condition. This statement is less naive than what it might seem at first sight; indeed, as it was shown in [3], the existence of a non-parity-invariant vacuum state would imply that the theory in presence of a parity-breaking term is ill-defined. On the other hand, if we take this assumption as a requirement, the conjecture becomes a full-fledged theorem, which states that, if the theory with any added  $P = -1$  external bosonic sources is well-defined, parity can not be spontaneously broken.

Let us see now an alternative way of getting the Vafa-Witten result, that makes use of the concept of the *p.d.f.* of a local operator. We do not intend to give an alternative proof in which we solve the issues arised in [3], but to show a different path to reach the same result, introducing the *p.d.f.* formalism. In our derivation we will assume that we can consistently define the quantum theory with a  $P$ -breaking local order parameter term.

Let be  $Y(A_\mu^a)$  a local operator constructed with Bose fields. The probability distribution function of this local operator in the effective gauge theory described by the partition function (2.2) is

$$P(c) = \left\langle \delta \left( c - \frac{1}{V} \int d^4x Y(x) \right) \right\rangle \quad (2.4)$$

where  $V$  is the space-time volume and the mean values are computed over all the Yang-Mills configurations using the integration measure defined in (2.2). Equivalently one can define its Fourier transform

$$P(q) = \int e^{iqc} P(c) dc \quad (2.5)$$

which, in our case, is given by the following expression

$$P(q) = \frac{\int [dA_\mu^a] \exp\{-S_{\text{YM}} + (\frac{iq}{V} - \lambda) \int d^4x Y(x) + \ln \det(\not{D} + m)\}}{\int [dA_\mu^a] \exp\{-S_{\text{YM}} - \lambda \int d^4x Y(x) + \ln \det(\not{D} + m)\}} \quad (2.6)$$

or, in a short notation:

$$P(q) = \left\langle \exp\left(\frac{iq}{V} \int d^4x Y(x)\right) \right\rangle \quad (2.7)$$

If we calculate  $P(c)$  for a local  $P$ -breaking operator in absence of a  $P$ -breaking term in the action of (2.2) ( $\lambda = 0$ ) we expect a Dirac  $\delta$  distribution centered at the origin if the vacuum state is non degenerate. On the contrary, if the vacuum state is degenerate. the expected form for  $P(c)$  is:

$$P(c) = \sum_{\alpha} w_{\alpha} \delta(c - c_{\alpha}) \quad (2.8)$$

where  $c_{\alpha}$  is the mean value of the local order parameter in the vacuum state  $\alpha$  and  $w_{\alpha}$  are positive real numbers which give us the probability of each vacuum state ( $\sum_{\alpha} w_{\alpha} = 1$ ).

If the only reason to have a degenerate vacuum is the spontaneous breaking of a discrete  $Z_2$  symmetry, we expect in the more standard case two symmetric vacuum states ( $\pm c_{\alpha}$ ) with the same weights  $w_{\alpha}$  ( in the most general case an even number of vacuum states with opposite values of  $c_{\alpha}$  ). The probability distribution function  $P(c)$  will be then the sum of two symmetric Dirac  $\delta$  with equal weights:

$$P(c) = \frac{1}{2} \delta(c - c_{\alpha}) + \frac{1}{2} \delta(c + c_{\alpha}) \quad (2.9)$$

and its Fourier transform

$$P(q) = \cos(qc_{\alpha}) \quad (2.10)$$

Now we are ready to apply this formalism to the case we are interested in. The relevant fact now, as in the Vafa-Witten paper, is the fact that the local  $P$ -breaking order parameter is a pure imaginary number  $Y(A_{\mu}^a) = iX(A_{\mu}^a)$ . In such a case we get for  $P(q)$ :

$$P(q) = \left\langle \exp\left(-\frac{q}{V} \int d^4x X(x)\right) \right\rangle \quad (2.11)$$

but the mean value of a real and positive quantity computed with a well defined probability distribution function (positive integration measure) can not be negative. Were the symmetry spontaneously broken we should get for  $P(q)$  a cosine function (2.10) (a sum of cosines in the most general case) which takes positive and negative values. Since this case is excluded,  $P(q)$  should be a constant function equal to 1, representing a symmetric vacuum state.

The generalization of the Vafa-Witten argumentation to fermionic bilinears upon integrating out the fermion fields, as suggested in [2], can not be carried out, as discussed in [9]. Indeed if we add a fermionic bilinear  $P = -1$  term,  $\lambda i \bar{\psi} \gamma_5 \psi$ , to the fermionic action, we get in the effective Yang-Mills theory an extra-term to the pure gauge action with a non-vanishing  $\lambda$ -dependent real part. Thus the contribution to the effective gauge theory

of the symmetry breaking term in the integrand of the partition function is no longer a pure phase factor and therefore how this term modifies the vacuum energy-density cannot be stated *a priori*. In fact there is a well known case, the Aoki phase [8] in lattice QCD with two degenerate flavours of Wilson fermions, in which notwithstanding the integration measure is non negative, Parity is spontaneously broken.

### 3. The *p.d.f.* for fermionic local operators

Our aim is to extend, as much as possible, the Vafa-Witten result for pure gluonic operators to fermion bilinear local operators. To this end we will make use, as in the previous section, of the *p.d.f.* of local operators constructed with Grassmann fields. Since this formalism is not, in our opinion, very standard, we want to devote this section to summarize the main results reported in [10] on this subject.

The motivation to develop this formalism was precisely the study of the vacuum invariance (non invariance), in quantum theories regularized on a space time lattice, under symmetry transformations which, as chiral, flavour or baryon symmetries, involve fermion fields. The numerical determination of the *p.d.f.* of the order parameter is a standard procedure when analyzing spontaneous symmetry breaking in spin systems or in quantum field theories with Bose fields. Indeed these degrees of freedom can be simulated directly on a computer. However in the numerical simulation of a Lattice Gauge Theory (LGT) with dynamical fermions the Grassmann fields, which at present can not be simulated in a computer, must be integrated analytically. Then even if the ground state of the theory is not invariant under the chiral, flavour or baryon symmetries, in the analytical procedure we integrate over all possible vacuum states, the order parameter vanishing always independently of the symmetry realization. This is the reason why the standard procedure to study, for instance, the chiral properties of the *QCD* vacuum in numerical simulations on a lattice is to add a mass term to the Euclidean action, which breaks chiral symmetry and to compute the chiral condensate  $\langle \bar{\psi}\psi \rangle$  as a function of the mass  $m$  performing, at the end, a numerical extrapolation to the chiral limit in order to see if  $\langle \bar{\psi}\psi \rangle$  takes a non-vanishing value in this limit.

Notwithstanding that Grassmann variables cannot be simulated in a computer, it was shown in [10] that an analysis of spontaneous symmetry breaking based on the use of the *p.d.f.* of fermion local operators, and therefore free from extrapolations in the symmetry breaking field, can also be done in *QFT* with fermion degrees of freedom.

The procedure is similar to the one used in the previous section for bosonic degrees of freedom. The starting point is to choose an order parameter for the desired symmetry  $O(\psi, \bar{\psi})$  (typically a fermion bilinear) and characterize each vacuum state  $\alpha$  by the expectation value  $c_\alpha$  of the order parameter in the  $\alpha$  state

$$c_\alpha = \frac{1}{V} \int \langle O(x) \rangle_\alpha d^4x \tag{3.1}$$

The *p.d.f.*  $P(c)$  of the order parameter will be given by

$$P(c) = \sum_\alpha w_\alpha \delta(c - c_\alpha) \tag{3.2}$$

which can also be written as [10]

$$P(c) = \left\langle \delta\left(\frac{1}{V} \int O(x)d^4x - c\right) \right\rangle \quad (3.3)$$

the mean value computed with the integration measure of the path integral formulation of the Quantum Theory.

The Fourier transform  $P(q) = \int e^{iqc}P(c)dc$  can be written, for the theory described by partition function (2.1), as

$$P(q) = \frac{\int [dA_\mu^a][d\bar{\psi}d\psi] \exp\{-S_{\text{YM}} + \bar{\psi}(\not{D} + m)\psi + \frac{iq}{V} \int d^4x O(x)\}}{\int [dA_\mu^a][d\bar{\psi}d\psi] \exp\{-S_{\text{YM}} + \bar{\psi}(\not{D} + m)\psi\}} \quad (3.4)$$

In the particular case in which  $O$  is a fermion bilinear of  $\bar{\psi}$  and  $\psi$

$$O(x) = \bar{\psi}(x)\tilde{O}\psi(x) \quad (3.5)$$

with  $\tilde{O}$  any matrix with Dirac, color and flavour indices, equation (3.4) becomes

$$P(q) = \frac{\int [dA_\mu^a][d\bar{\psi}d\psi] \exp\{-S_{\text{YM}} + \bar{\psi}(\not{D} + m + \frac{iq}{V}\tilde{O})\psi\}}{\int [dA_\mu^a][d\bar{\psi}d\psi] \exp\{-S_{\text{YM}} + \bar{\psi}(\not{D} + m)\psi\}} \quad (3.6)$$

Integrating out the fermion fields in (3.6) one gets

$$P(q) = \frac{\int [dA_\mu^a] e^{-S_{\text{YM}}} \det(\not{D} + m + \frac{iq}{V}\tilde{O})}{\int [dA_\mu^a] e^{-S_{\text{YM}}} \det(\not{D} + m)} \quad (3.7)$$

which can expressed also as the following mean value

$$P(q) = \left\langle \frac{\det(\not{D} + m + \frac{iq}{V}\tilde{O})}{\det(\not{D} + m)} \right\rangle \quad (3.8)$$

computed in the effective gauge theory with the integration measure

$$[dA_\mu^a] e^{-S_{\text{YM}}} \det(\not{D} + m)$$

The particular form expected for the *p.d.f.*  $P(c)$ ,  $P(q)$ , depends on the realization of the corresponding symmetry in the vacuum. A symmetric vacuum will give

$$P(c) = \delta(c) \quad (3.9)$$

$$P(q) = 1 \quad (3.10)$$

whereas, if we have for instance a  $U(1)$  symmetry which is spontaneously broken, the expected values for  $P(c)$  and  $P(q)$  are [10]

$$P(c) = [\pi(c_0^2 - c^2)^{1/2}]^{-1} \quad -c_0 < c < c_0 \quad (3.11)$$

$$P(c) = 0 \quad \text{otherwise} \quad (3.12)$$

$$P(q) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta e^{iqc_0 \cos \theta} \quad (3.13)$$

the last being the well known zeroth order Bessel function of the first kind,  $J_0(qc_0)$ .

In the simpler case in which a discrete  $Z(2)$  symmetry, as Parity, is spontaneously broken, the expected form is

$$P(c) = \frac{1}{2}\delta(c - c_0) + \frac{1}{2}\delta(c + c_0) \tag{3.14}$$

$$P(q) = \cos(qc_0) \tag{3.15}$$

or a sum of symmetric delta functions ( $P(c)$ ) and a sum of cosines ( $P(q)$ ) if there is an extra vacuum degeneracy.

This formalism has been successfully applied to the analysis of spontaneous chiral symmetry breaking in LGT with Kogut-Susskind fermions [10], as well as to find a new phase in finite baryon density two-colors QCD with diquark condensation and spontaneous breaking of baryon symmetry [11].

The analysis of the diquark condensation phase in finite density two-color QCD with four Kogut-Susskind flavours done in [11] showed also one of the potentialities of this approach. Indeed the order parameter used in the search for diquark condensation phase

$$\psi\tau_2\psi + \bar{\psi}\tau_2\bar{\psi}$$

when added to the  $SU(2)$  Lagrangian as an external source that breaks baryon symmetry, place us against a tedious problem, the well known sign problem. After integrating out the fermion fields we get the Pfaffian of a matrix and this Pfaffian is not positive definite. The standard approach to perform numerical analysis of spontaneous symmetry breaking, based on the computation of the condensate at non-vanishing external source, does not work because standard Monte Carlo techniques apply to systems with a well defined (positive definite) Boltzmann factor. The *p.d.f.* of the diquark condensate is computed, on the contrary, at vanishing external source, and the sign problem is absent in this case.

We are ready now to apply this approach in the next section to the analysis of spontaneous  $P$  breaking in vector-like theories, by investigating order parameters constructed from bilinears of the fermion field.

#### 4. Vacuum expectation values of $P = -1$ fermion bilinears

In this section we will show that all local bilinear  $P = -1$  gauge invariant operators of the form  $\bar{\psi}\tilde{O}\psi$ , with  $\tilde{O}$  a constant matrix with Dirac, color and flavour indices, should take a vanishing vacuum expectation value in any vector-like theory with  $N$  degenerate flavours.

Let us start with the one-flavour case since, as it will be shown, it is a special case. The standard hermitian local and gauge invariant  $P$  order parameter bilinear in the fermion fields is:

$$\bar{\psi}\tilde{O}\psi = i\bar{\psi}\gamma_5\psi$$

Now we can apply the result of the previous section, and in particular equation (3.8), which give us the *p.d.f.* of  $\tilde{O}$  in momentum space

$$P(q) = \left\langle \frac{\det(\not{D} + m + \frac{q}{V}\gamma_5)}{\det(\not{D} + m)} \right\rangle \tag{4.1}$$



The determinant of the Dirac operator in the denominator of (4.1) is positive definite, but the numerator of this expression, even if real, has not well defined sign. The final form for  $P(q)$  will depend crucially on the distribution of the real eigenvalues of  $\gamma_5(\not{D} + m)$ . Therefore we cannot say *a priori* whether  $P(q)$  will be the constant function  $P(q) = 1$  (symmetric vacuum) or a cosine function (spontaneously broken  $P$ ).

Let us go now to the  $N$  flavour case ( $N > 1$ ). The most general  $P = -1$  hermitian and gauge invariant local order parameters  $\bar{\psi}\tilde{O}\psi$  which can be constructed are:

$$i\bar{\psi}\gamma_5\psi, \quad i\bar{\psi}\gamma_5\bar{\tau}\psi \tag{4.2}$$

with  $\bar{\tau}$  any of the hermitian generators of the  $SU(N)$  flavour group. However, since flavour symmetry cannot be spontaneously broken in a vector-like theory [12], we will restrict our analysis to the flavour singlet case.

$$i\bar{\psi}\gamma_5\psi = i\bar{\psi}_u\gamma_5\psi_u + i\bar{\psi}_d\gamma_5\psi_d + i\bar{\psi}_s\gamma_5\psi_s + \dots \tag{4.3}$$

Let us assume that  $\langle i\bar{\psi}_u\gamma_5\psi_u \rangle = \pm c_0 \neq 0$ . Since flavour symmetry is not spontaneously broken we should have:

$$\langle i\bar{\psi}_u\gamma_5\psi_u \rangle = \langle i\bar{\psi}_d\gamma_5\psi_d \rangle = \langle i\bar{\psi}_s\gamma_5\psi_s \rangle = \dots \tag{4.4}$$

Thus the system will show two degenerate vacua with all the condensates oriented in the up (down) direction.

We want to remark that the “ferromagnetic” nature of these two vacua is imposed by the realization of flavour symmetry in the vacuum. Otherwise one could imagine also “antiferromagnetic” vacua with antiparallel condensates, or even more complex structures.

In vector-like gauge theories the interaction between different flavour is mediated by gluons. Then, assuming  $\langle i\bar{\psi}_u\gamma_5\psi_u \rangle \neq 0$  the impossibility to break flavour symmetry [12] could suggest that the gauge interaction would favor “ferromagnetic” vacua with parallel oriented condensates. However the actual dynamics can also be more complex. Indeed we will argue now that a non vanishing condensate  $\langle i\bar{\psi}_u\gamma_5\psi_u \rangle \neq 0$ , which would imply spontaneous breaking of  $P$ ,  $CP$ ,  $T$  and  $CT$ , can be excluded.

Let us apply equation (3.8) to the computation of the *p.d.f.* in momentum space  $P_{ud}(q)$  of  $i\bar{\psi}_u\gamma_5\psi_u + i\bar{\psi}_d\gamma_5\psi_d$ .

$$P_{ud}(q) = \left\langle \left( \frac{\det(\not{D} + m - \frac{q}{V}\gamma_5)}{\det(\not{D} + m)} \right)^2 \right\rangle \tag{4.5}$$

where  $\not{D} + m$  in (4.5) is the one flavour Dirac operator and the mean value is computed in the theory with  $N$ -degenerate flavours.

Equation (4.5) give us the mean value, computed with a positive definite integration measure, of a real non-negative quantity. Thus  $P_{ud}(q)$  will be a positive definite, or at least a non negative definite, quantity. But were  $\langle i\bar{\psi}_u\gamma_5\psi_u \rangle \neq 0$  we should expect a cosine function for  $P_{ud}(q)$  since  $i\bar{\psi}_u\gamma_5\psi_u$  and  $i\bar{\psi}_d\gamma_5\psi_d$  are enforced to take the same *v.e.v.* for flavour symmetry. Since the positivity of  $P_{ud}(q)$  excludes such a possibility, we can conclude that all the pseudo-scalar condensates  $i\bar{\psi}_f\gamma_5\psi_f$  take vanishing expectation values.

In order to give a simple illustration of some of these ideas, one can think of a physical system composed by two replicas of the two-dimensional Ising model. The spins  $s^a$  of replica  $a$  and  $s^b$  of replica  $b$  live on the sites of a two-dimensional lattice. They are coupled inside each replica with the standard nearest-neighbor ferromagnetic coupling, and we add an ultra-local interaction between  $s^a$  and  $s^b$  at the same site.

The Hamiltonian of this system can be written in the following way:

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} (s_i^a s_j^a + s_i^b s_j^b) - k \sum_i s_i^a s_i^b \quad (4.6)$$

This system has the standard  $Z(2)$   $P$  symmetry ( $\mathcal{H}$  is invariant under the change of sign of all spins) plus an extra  $Z(2)$  symmetry, the replica symmetry, as a consequence of the invariance of  $\mathcal{H}$  under the permutations of the  $a$  and  $b$  indices.

Let us neglect for a while the replica-replica interaction ( $k = 0$  in (4.6)) and choose  $F = \frac{J}{KT} > 0.44$  (ferromagnetic phase). If we denote the magnetization per site of replica  $a$  and  $b$  as  $m_a$  and  $m_b$ , we can construct two order parameters:  $\langle m_a - m_b \rangle$  and  $\langle m_a + m_b \rangle$ . Both are  $P = -1$  order parameters and the first one is also order parameter for the  $Z(2)$  replica symmetry.

Under the previous assumed conditions ( $F > 0.44$ ,  $k = 0$ ) we have four degenerate vacua ( $++$ ,  $--$ ,  $+-$ ,  $-+$ ) characterized by the following values of the order parameters:

$$\langle m_a + m_b \rangle_{++} = 2m_0 \quad \langle m_a - m_b \rangle_{++} = 0 \quad (4.7)$$

$$\langle m_a + m_b \rangle_{--} = -2m_0 \quad \langle m_a - m_b \rangle_{--} = 0 \quad (4.8)$$

$$\langle m_a + m_b \rangle_{+-} = 0 \quad \langle m_a - m_b \rangle_{+-} = 2m_0 \quad (4.9)$$

$$\langle m_a + m_b \rangle_{-+} = 0 \quad \langle m_a - m_b \rangle_{-+} = -2m_0 \quad (4.10)$$

If we switch on now the replica-replica interaction ( $k \neq 0$ ) we expect that the vacuum degeneracy will be reduced. Indeed  $k > 0$  values will enforce a parallel orientation of spins of different replicas, whereas  $k < 0$  will favor an anti-parallel orientation. In the first case we will get two “ferromagnetic” degenerate vacua preserving the replica symmetry. In the second case we get two “antiferromagnetic” degenerate vacua with spontaneous parity and replica breaking.

What we can learn from this simple model is that a small replica-replica interaction is enough to break some of the vacuum degeneracy. Then “ferromagnetic” or “antiferromagnetic” degenerate vacua are selected, depending on the ferro-antiferromagnetic character of the replica-replica interaction.

## 5. Conclusions

We have reconsidered in this paper the old aim of trying to understand if the observed realization of discrete symmetries as Parity or  $CP$  in the  $QCD$  vacuum can be satisfied from first principles. Although this aim is at the moment too ambitious, we have shown how under the appropriate plausible assumptions implicitly done in [2], all parity and  $CP$  odd operators constructed from fermion bilinears of the form  $\bar{\psi} \tilde{O} \psi$  should take a vanishing

vacuum expectation value in a vector-like theory with  $N$  degenerate flavours ( $N > 1$ ). In our analysis the Vafa-Witten theorem on the impossibility to break spontaneously the flavour symmetry in a vector-like theory [12] plays a fundamental role. Indeed our result does not apply to lattice  $QCD$  with two degenerate Wilson fermions, where a phase with spontaneous Parity breaking has been found [8]. It follows because, notwithstanding the integration measure is non negative, the Vafa-Witten theorem on the impossibility of breaking spontaneously a vector symmetry in a vector-like theory is not applicable to this case, for the Wilson quark propagator is not bounded in an arbitrary background gauge field. Indeed flavour symmetry is spontaneously broken in the Aoki phase.

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